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An Operator Calculus for Related Partial Differential Equations

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1. INTRODUCTION

In a recent paper [1] the authors gave a result which related, through Laplace transforms, the solutions of initial-boundary value problems based on the related equations $D_t u(x, t) = P(x, D)\, u(x, t)$ and $D_t^2 v(x, t) = P(x, D)\, v(x, t)$, where $D_t = \partial/\partial t$, $x = (x_1, x_2, \ldots, x_n)$, and P is a finite-order linear partial differential operator which does not depend on t. In another paper [2], we gave several examples exploiting the relationships thus established between the wave, damped wave, Klein-Gordon equations, and the heat equation. The basic relation which connects solutions of boundary value problems satisfying $u(x, 0) = \phi(x)$ and $v(x, 0) = 0$, $v_t(x, 0) = \phi(x)$ is

$$v(x, t) = \Gamma(\tfrac{3}{2})\, \mathscr{L}_s^{-1} \left\{ s^{-3/2} u\left(x, \frac{1}{4s}\right)\right\}_{s\to t^2}, \tag{1.1}$$

where the symbolism denotes the fact that s is the variable of the Laplace transform and the variable in the inverted function is replaced by t^2. Obviously, at the heart of the matter is some relationship between the operators D_t and D_t^2. The solution operator associated with the equation $D_t u = Pu$ is e^{tP}. On the other hand, if we solve the equation $D_t^2 v = Pv$, as if P were a constant, subject to the initial conditions $v(0) = 0$, $v_t(0) = 1$, we obtain

$$v = \frac{\sinh t\sqrt{P}}{\sqrt{P}}.$$

We take this to be the solution operator of the second problem. The solution operators are related by the Laplace transform in the following way:

$$\frac{\sinh t\sqrt{P}}{\sqrt{P}} = \Gamma(\tfrac{3}{2})\, \mathscr{L}_s^{-1}\{s^{-3/2} e^{P/4s}\}_{s\to t^2} \tag{1.2}$$

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Operating formally on the data function $\phi(x)$ on both sides gives us the relationship (1.1).

This suggests the following method for relating the equations

$$0_1(D_t)\, u(x, t) = P(x, D)\, u(x, t) \qquad \text{and} \qquad 0_2(D_t)\, v(x, t) = P(x, D)\, v(x, t).$$

Find the appropriate solution operators from the ordinary differential equations $0_1(D_t)\, u = Pu$ and $0_2(D_t)\, v = Pv$, subject to initial conditions related to those for the corresponding partial differential equations. Connect the solution operators through the Laplace transform (or other transform) and hence obtain the corresponding relation between the solutions of the related partial differential equations. We do not attempt to prove that the method works, in general, because we have found that once the proper relationship between solutions is obtained it is a simple matter to establish the result rigorously for general operators $P(x, D)$ and general boundary conditions. The results are valid for generalized solutions. We give one such proof in Section 4, for illustrative purposes.

In this paper, we shall illustrate the operator method for relating solutions of partial differential equations by giving examples of several related problems. One of these will involve the Euler-Poisson-Darboux equation, which can be simply related to the heat equation. If a space variable ranges from zero to infinity, as in the case of the semi-infinite heat conducting rod, then it may be possible to make a relation through a space variable. In this case, we relate the heat equation to a first-order equation. We also relate on a space variable to find the transverse vibrations of a semi-infinite beam.

2. Different Initial Conditions

If in the above example, we change the initial conditions in the second problem to $v(x, 0) = \phi(x)$, $v_t(x, 0) = 0$, then we seek the solution operator for the equation $D_t^2 v = Pv$, subject to $v(0) = 1$, $v_t(0) = 0$. The solution operator is $\cosh t\sqrt{P}$ and the required relation is

$$\cosh t\sqrt{P} = t\Gamma(\tfrac{1}{2})\, \mathscr{L}_s^{-1}\{s^{-1/2}e^{P/4s}\}_{s\to t^2}\,, \tag{2.1}$$

giving us

$$v(x, t) = t\Gamma(\tfrac{1}{2})\, \mathscr{L}_s^{-1}\left\{s^{-1/2}u\left(x, \frac{1}{4s}\right)\right\}_{s\to t^2}. \tag{2.2}$$

This formula can be verified as follows. One can obtain the solution given by formula (2.2) by finding a solution $V(x, t)$ as given in (1.1) for the initial conditions $V(x, 0) = 0$, $V_t(x, 0) = \phi(x)$, and then by differentiating $V(x, t)$ with respect to t. Let $\tau = t^2$ be the variable in the inverse transform. Then

$$v(x, t) = \frac{\partial V}{\partial t} = \frac{\partial V}{\partial \tau}\frac{d\tau}{dt} = t\Gamma(\tfrac{1}{2})\, \mathscr{L}_s^{-1}\left\{s^{-1/2}u\left(x, \frac{1}{4s}\right)\right\}_{s\to t^2}.$$

3. The Backward Heat Equation

If $P(x, D) = -\Delta_n$, the n-dimensional Laplacian, then $D_t u = -\Delta_n u$ is the backward heat equation. The related equation $D_t^2 v = -\Delta_n v$ is the $(n+1)$-dimensional Laplace equation. The solution operator associated with $D_t^2 v = -\Delta_n v$, $v(0) = 1$, $v_t(0) = 0$ is $\cos t\sqrt{\Delta_n}$. The solution operator associated with $D_t^2 v = -\Delta_n v$, $v(0) = 0$, $v_t(0) = 1$ is $(\sin t\sqrt{\Delta_n}/\sqrt{\Delta_n})$. The corresponding transforms are

$$\cos t\sqrt{\Delta_n} = t\Gamma(\tfrac{1}{2})\, \mathscr{L}_s^{-1}\{s^{-1/2}e^{-\Delta_n/4s}\}_{s\to t^2} \tag{2.3}$$

$$\frac{\sin t\sqrt{\Delta_n}}{\sqrt{\Delta_n}} = \Gamma(\tfrac{3}{2})\, \mathscr{L}_s^{-1}\{s^{-3/2}e^{-\Delta_n/4s}\}_{s\to t^2}, \tag{2.4}$$

which lead to the relationships:

$$v(x, t) = t\Gamma(\tfrac{1}{2})\, \mathscr{L}_s^{-1}\left\{s^{-1/2}u\left(x, \frac{1}{4s}\right)\right\}_{s\to t^2} \tag{2.5}$$

for the initial conditions $v(x, 0) = \phi(x)$, $v_t(x, 0) = 0$, and

$$V(x, t) = \Gamma(\tfrac{3}{2})\, \mathscr{L}_s^{-1}\left\{s^{-3/2}u\left(x, \frac{1}{4s}\right)\right\}_{s\to t^2} \tag{2.6}$$

for the initial conditions $V(x, 0) = 0$, $V_t(x, 0) = \phi(x)$, where in each case $u(x, t)$ is the solution of the backward heat equation with initial data $u(x, 0) = \phi(x)$.

It is well known that the initial value problems for Laplace's equation are not well-posed. But then so is the case with the backward heat equation. The relationship we have established indicates that the difficulty is not inherently one of classification of the partial differential equation. Also the relationship gives us the means of studying the question of well-posedness of the one problem in terms of the other.

4. The Euler-Poisson-Darboux Equation [3], [4]

In this section let us consider the related problems $D_t u = \Delta_n u$, $u(x, 0) = \phi(x)$ and $D_t^2 v + (a/t) D_t v = \Delta_n v$, $a \geqslant 1$, $v(x, 0) = \phi(x)$, $v_t(x, 0) = 0$. We find the solution operator for the equation $D_t^2 v + (a/t) v_t = \Delta_n v$, subject to initial conditions $v(0) = 1$, $v_t(0) = 0$. The operator is

$$\frac{2^{(a-1)/2} \Gamma\left(\frac{a+1}{2}\right) t^{(1-a)/2}}{\Delta_n^{(a-1)/4}} I_{(a-1)/2}(t\Delta_n^{1/2}),$$

where I_ν is a modified Bessel function of order ν. The relationship between the operators is the following

$$\frac{2^{(a-1)/2} \Gamma\left(\frac{a+1}{2}\right) t^{(1-a)/2}}{\Delta_n^{(a-1)/4}} I_{(a-1)/2}(t\Delta_n^{1/2})$$

$$= t^{1-a} \mathscr{L}_s^{-1} \left\{ \Gamma\left(\frac{a+1}{2}\right) s^{-(a+1)/2} e^{\Delta_n/4s} \right\}_{s \to t^2} \tag{4.1}$$

This leads to the relationship between solutions,

$$v(x, t) = t^{1-a} \mathscr{L}_s^{-1} \left\{ \Gamma\left(\frac{a+1}{2}\right) s^{-(a+1)/2} u\left(x, \frac{1}{4s}\right) \right\}_{s \to t^2} \tag{4.2}$$

We prove the following theorem which establishes the relationship (4.2), under more general conditions.

Theorem. *Let* $x = (x_1, x_2, \ldots, x_n)$ *and* $D = (D_1, D_2, \ldots, D_n)$ *where* $D_i f(x) = \partial f / \partial x_i$. Let $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$ *and let*

$$P(x, D) = \sum_{0 \leqslant |\alpha| \leqslant m} a_\alpha(x) D^\alpha,$$

where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ *and the* $a_\alpha(x)$ *are given functions of* x. *Let* $S(x) = 0$ *denote a cylindrical surface in the* (x, t) *space and* $B(x, D)$ *a non-tangential boundary operator whose domain is the manifold* $S(x) = 0$. *Let* $u(x, t)$ *be the solution of* $\partial u / \partial t = P(x, D) u$, $t > 0$, $u(x, 0) = \phi(x)$, $B(x, D) u(x, t) = f(x, t)$, $x \in S$, $t > 0$; *and let* $v(x, t)$ *be the solution of* $\partial^2 v / \partial t^2 + (a/t) \partial v / \partial t = P(x, D) v$, $a \geqslant 1$, $t > 0$, $v(x, 0) = \phi(x)$, $v_t(x, 0) = 0$, $B(x, D) v(x, t) = g(x, t)$, $x \in S$, $t > 0$.

If $B(x, D)\,\phi(x)$ *vanishes on* $S(x) = 0$, $P(x, D)\,\phi(x)$ *is continuous and*

$$g(x, t) = t^{1-a}\mathscr{L}_s^{-1}\left\{\Gamma\left(\frac{a+1}{2}\right) s^{-(a+1)/2} f\left(x, \frac{1}{4s}\right)\right\}_{s\to t^2}$$

then

$$v(x, t) = t^{1-a}\mathscr{L}_s^{-1}\left\{\Gamma\left(\frac{a+1}{2}\right) s^{-(a+1)/2} u\left(x, \frac{1}{4s}\right)\right\}_{s\to t^2},$$

provided the inverse transforms exist.

PROOF. We first introduce changes of variables $u(x, t) = u^*(x, t) + \phi(x)$ and $v(x, t) = v^*(x, t) + \phi(x)$. Then $u_t^* = P(x, D)\,u^* + P(x, D)\,\phi$, $u^*(x, 0) = 0$ $B(x, D)\,u^* = f(x, t)$, $x \in S$; and $v_{tt}^* + (a/t)\,v_t^* = P(x, D)\,v^* + P(x, D)\,\phi$, $v^*(x, 0) = v_t^*(x, 0) = 0$, $B(x, D)\,v^* = g(x, t)$, $x \in S$. Next, introduce the variable $V(x, t) = t^{a-1}v^*(x, t)$. This gives

$$V_{tt} + \left(\frac{2-a}{t}\right) V_t = P(x, D)\,V + t^{a-1}P(x, D)\,\phi;$$

$$V(x, 0) = V_t(x, 0) = 0, \qquad B(x, D)\,V = t^{a-1}g(x, t), \qquad x \in S.$$

Next we introduce the variable $\tau = t^2$, which results in

$$4\tau V_{\tau\tau} + (6 - 2a)\,V_\tau = P(x, D)\,V + \tau^{(a-1)/2}\,P(x, D)\,\phi$$

with

$$V(x, 0) = V_\tau(x, 0) = 0, \qquad B(x, D)\,V = \tau^{(a-1)/2}g(x, \tau^{1/2}), \qquad x \in S.$$

Now introduce the Laplace transform $\bar{V}(x, s)$ of $V(x, \tau^{1/2})$. This satisfies

$$4s^2\frac{\partial \bar{V}}{\partial s} + s(2 + a)\,\bar{V} + P(x, D)\,\bar{V} + \frac{\Gamma\left(\frac{a+1}{2}\right)}{s^{(a+1)/2}}\,P(x, D)\,\phi = 0$$

or

$$4s^2\frac{\partial}{\partial s}\left(\frac{s^{(a+1)/2}\bar{V}}{\Gamma\left(\frac{a+1}{2}\right)}\right) + P(x, D)\left(\frac{s^{(a+1)/2}\bar{V}}{\Gamma\left(\frac{a+1}{2}\right)}\right) + P(x, D)\,\phi = 0$$

with

$$B(x, D)\left(\frac{s^{(a+1)/2}\bar{V}}{\Gamma\left(\frac{a+1}{2}\right)}\right) = \frac{s^{(a+1)/2}\bar{G}(x, s)}{\Gamma\left(\frac{a+1}{2}\right)}, \qquad x \in S,$$

where $\bar{G}(x, s)$ is the Laplace transform of $\tau^{(a-1)/2} g(x, \tau^{1/2})$. In the other problem, we introduce the variable $s = 1/4t$, and we have

$$4s^2 \frac{\partial}{\partial s} u^* \left(x, \frac{1}{4s}\right) + P(x, D)\, u^* \left(x, \frac{1}{4s}\right) + P(x, D)\, \phi = 0$$

with

$$B(x, D)\, u^* \left(x, \frac{1}{4s}\right) = f\left(x, \frac{1}{4s}\right), \qquad x \in S.$$

A comparison of the transformed problems show that the functions $u^*(x, (1/4s))$ and $s^{(a+1)/2}\bar{V}(x, s)/\Gamma[(a+1)/2]$ satisfy the same differential equation and boundary conditions provided

$$f\left(x, \frac{1}{4s}\right) = \frac{s^{(a+1)/2}}{\Gamma\left(\frac{a+1}{2}\right)} \bar{G}(x, s)$$

$$\lim_{s\to\infty} s^{(a+1)/2}\bar{V}(x, s) = 0$$

Imposing these conditions and inverting the transforms gives us the desired result.

There is an obvious corollary which gives $u(x, t)$ in terms of $v(x, t)$ as follows:

$$u(x, t) = \frac{2}{(4t)^{(a+1)/2}\, \Gamma\left(\frac{a+1}{2}\right)} \int_0^\infty \xi^a e^{-(\xi^2/4t)} v(x, \xi)\, d\xi \tag{4.3}$$

provided the boundary conditions are properly related.

5. The Fundamental Solution of the Euler-Poisson-Darboux Equation

If we take

$$\phi(x) = \prod_{i=1}^{n} \delta(x_i - \xi_i),$$

where $\delta(z)$ is the Dirac distribution, the fundamental solution of the heat equation is

$$u(x, t) = \frac{1}{(4\pi t)^{n/2}}\, e^{-b_n^2(x,\xi)/4t} \tag{5.1}$$

with

$$b_n^2(x, \xi) = \sum_{i=1}^{n} (x_i - \xi_i)^2.$$

The corresponding solution of the Euler-Poisson-Darboux equation is

$$v(x, t) = \frac{t^{1-a}\Gamma\left(\frac{a+1}{2}\right)}{\pi^{n/2}} \mathscr{L}_s^{-1}\{s^{(n-a-1)/2}e^{-b_n^2 s}\}_{s\to t^2}. \tag{5.2}$$

If $a > n - 1$

$$\mathscr{L}_s^{-1}\{s^{(n-a-1)/2}e^{-b_n^2 s}\}_{s\to t^2} = \begin{cases} 0, & \text{if} \quad t^2 \leqslant b_n^2 \\ \dfrac{(t^2 - b_n^2)^{(a-1-n)/2}}{\Gamma\left(\frac{a+1-n}{2}\right)}, & \text{if} \quad t^2 > b_n^2. \end{cases}$$

In this case, the general solution of the Euler-Poisson-Darboux equation is

$$v(x, t) = \frac{t^{1-a}\Gamma\left(\frac{a+1}{2}\right)}{\pi^{n/2}\Gamma\left(\frac{a+1-n}{2}\right)} \times \int_{B(x,t)} \frac{\phi(\xi_1, \xi_2, \ldots, \xi_n)}{\left[t^2 - \sum_{i=1}^{n}(x_i - \xi_i)^2\right]^{(n+1-a)/2}} d\xi_1 \cdots d\xi_n, \tag{5.3}$$

where $B(x, t)$ is the ball $\sum_{i=1}^{n} (x_i - \xi_i)^2 \leqslant t^2$.

If $a \leqslant n - 1$, the inverse transform is not easy to compute unless $\frac{1}{2}(n - a - 1) = k$, a non-negative integer. In this case,

$$\mathscr{L}_s^{-1}\{s^k e^{-b_n^2 s}\}_{s\to t^2} = \delta^{(k)}(t^2 - b_n^2),$$

where $\delta^{(k)}(z)$ is the kth derivative of the delta distribution. If $S(x; t)$ is the surface of the sphere with center at x and radius t, then clearly the solution $v(x, t)$ at time t depends only on the values of $\phi(x)$ on $S(x; t)$. Thus the solution exhibits the clean-cut wave effect. If $a = n - 1$, $k = 0$, then using the result

$$\delta(t^2 - b^2) = \frac{1}{2t}\delta(t - b)$$

which was proved in [2], the general solution is (see [5])

$$v(x, t) = \frac{t^{2-n}\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}} \int_{S(x,t)} \frac{\phi(\xi_1, \xi_2, \ldots, \xi_n)}{t} \, dS_n , \tag{5.4}$$

where dS_n denotes the surface element on $S(x; t)$.

6. Temperature in a Semi-infinite Rod

An example of a problem where we can relate equations on a space variable is the problem of determining the temperature in a semi-infinite heat conducting rod. The problem is $D_x^2 v(x, t) = D_t v(x, t)$, $v(0, t) = f(t)$, $v(x, 0) = 0$. The related problem is $D_x u(x, t) = D_t u(x, t)$, $u(0, t) = f(t)$, $u(x, 0) = 0$. The solution of this problem is $u(x, t) = e(t - x) f(t + x)$, where $e(z)$ is the Heaviside step function. However,

$$s^{-1/2} e\left(t - \frac{1}{4s}\right) f\left(t + \frac{1}{4s}\right)$$

is not a Laplace transform, and if it were the solution of the original problem obtained by inversion would satisfy $v_x(0, t) = 0$, which is not the case in our problem. We therefore take a different approach. We attempt to find a solution by superimposing the solutions of two problems:

$$P_1 \begin{cases} v_{xx}^{(1)} = v_t^{(1)} \\ v^{(1)}(0, t) = f(t), \qquad v_x^{(1)}(0, t) = 0 \end{cases}$$

and

$$P_2 \begin{cases} v_{xx}^{(2)} = v_t^{(2)} \\ v^{(2)}(0, t) = 0, \qquad v_x^{(2)}(0, t) = g(t). \end{cases}$$

The functions $f(t)$ and $g(t)$ are to be differentiable. The function $g(t)$ is determined so that $v(x, 0) = v^{(1)}(x, 0) + v^{(2)}(x, 0) = 0$. The related problems are

$$P_1' \begin{cases} u_x^{(1)} = u_t^{(1)} \\ u^{(1)}(0, t) = f(t) \end{cases}$$

$$P_2' \begin{cases} u_x^{(2)} = u_t^{(2)} \\ u^{(2)}(0, t) = g(t) \end{cases}$$

and they have the solutions

$$u^{(1)}(x, t) = f(x + t)$$
$$u^{(2)}(x, t) = g(x + t).$$

The corresponding problems have solutions

$$v^{(1)}(x, t) = x\Gamma(\tfrac{1}{2})\, \mathscr{L}_s^{-1} \left\{ s^{-1/2} f\left(\frac{1}{4s} + t\right) \right\}_{s \to x^2}$$

$$v^{(2)}(x, t) = \Gamma(\tfrac{3}{2})\, \mathscr{L}_s^{-1} \left\{ s^{-3/2} g\left(\frac{1}{4s} + t\right) \right\}_{s \to x^2}.$$

The condition $v^{(1)}(x, 0) + v^{(2)}(x, 0) = 0$ leads to

$$\mathscr{L}_s^{-1} \left\{ s^{-3/2} g\left(\frac{1}{4s}\right) \right\}_{s \to x^2} = -\, 2x \mathscr{L}_s^{-1} \left\{ s^{-1/2} f\left(\frac{1}{4s}\right) \right\}_{s \to x^2}.$$

Introducing the variable $\tau = x^2$, gives us

$$\mathscr{L}_s^{-1} \left\{ s^{-3/2} g\left(\frac{1}{4s}\right) \right\}_{s \to \tau} = -\, 2\sqrt{\tau}\, \mathscr{L}_s^{-1} \left\{ s^{-1/2} f\left(\frac{1}{4s}\right) \right\}_{s \to \tau}.$$

Let

$$F(\tau) = \mathscr{L}_s^{-1} \left\{ s^{-1/2} f\left(\frac{1}{4s}\right) \right\}_{s \to \tau}.$$

Then (compare with [6], pp. 525, 526)

$$g(t) = -\, \tfrac{1}{2}\, t^{-3/2} \int_0^\infty \eta^2 e^{-\eta^2/4t} F(\eta^2)\, d\eta.$$

As an example, let us take $f(t) = t$. Then $g(t) = -\,(2/\sqrt{\pi})t^{1/2}$. Hence, we have

$$v^{(1)}(x, t) = x\Gamma(\tfrac{1}{2})\, \mathscr{L}_s^{-1} \left\{ s^{-1/2} \left(\frac{1}{4s} + t\right) \right\}_{s \to x^2} = \frac{x^2}{2} + t$$

$$v^{(2)}(x, t) = -\, \mathscr{L}_s^{-1} \left\{ s^{-3/2} \left(\frac{1}{4s} + t\right)^{1/2} \right\}_{s \to x^2}$$

$$= -\sqrt{\frac{4t}{\pi}} \int_0^x e^{-\eta^2/4t}\, d\eta - \frac{1}{\sqrt{4\pi t}} \int_0^{x^2} \int_0^\xi e^{-\eta^2/4t}\, d\eta\, d\xi.$$

Then

$$v(x, t) = \frac{x^2}{2} + t - \sqrt{\frac{4t}{\pi}} \int_0^x e^{-\eta^2/4t}\, d\eta - \frac{1}{\sqrt{4\pi t}} \int_0^{x^2} \int_0^\xi e^{-\eta^2/4t}\, d\eta\, d\xi.$$

Compare this with [7], pp. 289-291.

7. Transverse Vibrations of a Semi-infinite Beam

The partial differential equation for the transverse vibrations of a uniform beam is $D_t^2 v = -D_x^4 v$. If we relate on the time variable, the related equation will be $D_t u = -D_x^4 u$. On the other hand, if we relate on the space variable, the related equation could be, for example, Laplace's equation $D_x^2 u = -D_t^2 u$. Using the method of operators we shall relate the problems

$$P_1 \begin{cases} D_x^4 v = Pv \\ v(0, t) = v_x(0, t) = v_{xx}(0, t) = 0 \\ v_{xxx}(0, t) = \phi(t) \end{cases}$$

$$P_2 \begin{cases} D_x^2 u = Pu \\ u(0, t) = 0 \\ u_x(0, t) = \phi(t). \end{cases}$$

To obtain the solution operator for problem P_2 we consider the equation $D^2 u = Pu$, subject to $u(0) = 0$, $u_x(0) = 1$. The solution operator is

$$\frac{\sinh x\sqrt{P}}{\sqrt{P}}.$$

For the solution operator for problem P_1, we solve $D^4 v = Pv$, subject to $v(0) = v_x(0) = v_{xx}(0) = 0$, $v_{xxx}(0) = 1$. The result is

$$\frac{\sinh xP^{1/4} - \sin xP^{1/4}}{2P^{3/4}}.$$

The relationship between the operators is

$$\frac{\sinh xP^{1/4} - \sin xP^{1/4}}{2P^{3/4}} = \Gamma(\tfrac{3}{2})\, \mathscr{L}_s^{-1} \left\{ \frac{s^{-3/2} \sinh \sqrt{P}/4s}{\sqrt{P}} \right\}_{s \to x^2} \tag{7.1}$$

and the relationship between solutions is

$$v(x, t) = \Gamma(\tfrac{3}{2})\, \mathscr{L}_s^{-1} \left\{ s^{-3/2} u\left(\frac{1}{4s}, t\right) \right\}_{s \to x^2}. \tag{7.2}$$

Notice that $V(x, t) = v_x(x, t)$ satisfies the differential equation $D_x^4 V = PV$ and $V(0, t) = V_x(0, t) = 0$, $V_{xx}(0, t) = \phi(t)$, $V_{xxx}(0, t) = 0$. Therefore, $V(x, t)$ is related to the solution of problem P_2 as follows:

$$V(x, t) = x\Gamma(\tfrac{1}{2})\, \mathscr{L}_s^{-1} \left\{ s^{-1/2} u\left(\frac{1}{4s}, t\right) \right\}_{s \to x^2}. \tag{7.3}$$

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